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STRUCTURE SPACES OF SEMIGROUPS OF CONTINUOUS FUNCTIONS

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Abstract. In a previous paper, we associated a topological space with each left ideal of a semigroup. Here, we determine this space when the semigroup under consideration is the semigroup of all continuous selfmaps of any space belonging to a fairly extensive class of topological spaces and the left ideal is taken to be the kernel of the semigroup.

1. **Introduction and statement of main theorem.** In [3], we associated with each left ideal Z of a semigroup T, a structure space which we denoted by $\mathscr{U}(T, Z)$. This space is formed as follows: a nonempty subset A of $T\times Z$ is a *bond* if for any finite subset $\{(t_i, z_i)\}_{i=1}^N \subseteq A$, the system of equations $\{t_i x = z_i\}_{i=1}^N$ has a common solution x in Z. An *ultrabond* is a bond which is not properly contained in any other bond. $\mathscr{U}(T, Z)$ is then defined to be the family of all such ultrabonds. $\mathscr{U}(T, Z) \neq \varnothing$ since the existence of a bond is immediate and by Zorn's Lemma, every bond is contained in ultrabond. We topologize $\mathscr{U}(T, Z)$ in the following manner: for each $(t, z) \in T\times Z$, let

$$H(t, z) = \{A \in \mathscr{U}(T, Z) : (t, z) \in A\}$$

and take $\{H(t, z) : (t, z) \in T\times Z\}$ to be a subbasis for the closed subsets of $\mathscr{U}(T, Z)$. For some general facts about such spaces, one should consult [3, pp. 319–324]. In the event Z is the kernel (minimal two-sided ideal) of T_1, we refer to $\mathscr{U}(T, Z)$ as the *$\mathscr{K}$-structure space of $\mathscr{T}$* and denote it simply by $\mathscr{U}(T)$.

Before we can state the Main Theorem, we need to recall some facts about E-compact spaces which were introduced by Engleking and Mrówka in [1]. We use the terminology and notation adopted in [4] which gives a rather extensive account of the theory. Let E be any Hausdorff space. A space X is E-completely regular if it is homeomorphic to a subset of some cartesian product of copies of E and it is E-compact if it is homeomorphic to a closed subset of such a product. We will refer to an E-compact space Y which contains X as a dense subspace as an E-compactification of X. An important fact about E-compactifications is that if E is compact in the usual sense, then each E-completely regular space X has a

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largest E-compactification $\beta_E X$ in the sense that any other E-compactification of X is a continuous image of $\beta_E X$ under a map which keeps the points of X fixed. This is an immediate consequence of Theorem 4.14 of [4, p. 177]. Before stating our Main Theorem, we introduce one additional concept.

DEFINITION. Let E be Hausdorff. A space X is E-separated if it is Hausdorff and for each pair H and K of disjoint nonempty closed subsets of X, there exists a continuous function f from X into E and two distinct points p and q of E such that $f(x)=p$ for $x \in H$ and $f(x)=q$ for $x \in K$.

Evidently, a space X is normal if and only if it is $\mathscr{I}$-separated where $\mathscr{I}$ denotes the closed unit interval. Furthermore, it follows rather quickly from Theorem 2.1 [4, p. 165] that every E-separated space is also E-completely regular. We are now in a position to state our main result which concerns the $\mathscr{K}$-structure space of the semigroup $S(X)$ of all continuous selfmaps of X where the binary operation is ordinary composition.

MAIN THEOREM. *Suppose that E is compact and that X is E-separated and contains a copy of E. Then the $\mathscr{K}$-structure space of $S(X)$ is, in fact, $\beta_E X$ the largest E-compactification of X.*

2. **Proof of the main theorem and two corollaries.** Since the proof will rely rather heavily in several instances upon Theorem (1.10) of [3, p. 322], we begin by discussing some concepts which are relevant to that result. Once again T is a semigroup and Z is a left ideal of T. If T has a left identity, then for each $v \in Z$, $A_v=\{(t, tv) : t \in T\}$ is an ultrabond [3, Lemma (1.2), p. 320] and the set of all ultrabonds of this form is denoted by $\mathscr{R}(T, Z)$ and is referred to as the realization of Z. This is a subspace of $\mathscr{U}(T, Z)$ and may well be a proper subspace. Now to each element $a \in T$, one can associate in a very natural way an element $\mathfrak{f}_a$ in $S(\mathscr{R}(T, Z))$ the semigroup, under composition, of all continuous selfmaps of $\mathscr{R}(T, Z)$. The mapping $\mathfrak{f}_a$ is defined by $\mathfrak{f}_a(A_v)=A_{av}$ for each $A_v \in \mathscr{R}(T, Z)$. Theorem (1.8) of [3, p. 321] asserts, among other things, that $\mathfrak{f}_a$ is continuous. Now we are in a position to extract the portion of Theorem (1.10) of [3, p. 322] which we need here. It is as follows:

(1) If the pair (T, Z) is admissible and T has a left identity then $\mathscr{U}(T, Z)$ is a Hausdorff compactification of $\mathscr{R}(T, Z)$ and each $\mathfrak{f}_a$ in $S(\mathscr{R}(T, Z))$ has a unique extension to a function $\mathfrak{f}_a^E$ in $S(\mathscr{U}(T, Z))$. The pair (T, Z) is defined to be admissible [3, Definition (1.6), p. 320] if

(2) A is an ultrabond and $A \in \mathscr{C}H(t_1, z_1)$ (the complement of $H(t_1, z_1)$ in $\mathscr{U}(T, Z)$) then there exist (t_2, z_2) and (t_3, z_3) in $T \times Z$ such that

$$A \in \mathscr{C}H(t_2, z_2) \subseteq H(t_3, z_3) \subseteq \mathscr{C}H(t_1, z_1).$$

Now suppose we turn our attention to the semigroup $S(X)$ where X satisfies the conditions stated in the Main Theorem. For any point $p \in X$, we denote by $\langle p \rangle$ the constant function which maps each point of X into p. One easily verifies that the kernel $K(X)$ of $S(X)$ is precisely the family of all constant function on X. Furthermore, it is not difficult to show that

(3) For any $f \in S(X)$ and $\langle y \rangle$, $\langle z \rangle$ in $K(X)$, $f \circ \langle y \rangle = \langle z \rangle$ if and only if $f(y) = z$.

(4) A subset A of $S(X) \times K(X)$ is a bond if and only if $\{f^{-1}(z) : (f, \langle z \rangle) \in A\}$ has the finite intersection property.

(5) A is an ultrabond if and only if $(f, \langle z \rangle) \notin A$ implies $f^{-1}(z) \cap g_1^{-1}(y_1) \cap \cdots \cap g_N^{-1}(y_N) = \varnothing$ for some finite subfamily $\{(g_i, \langle y_i \rangle)\}_{i=1}^N$ of A.

We next want to observe that the pair $(S(X), K(X))$ is admissible. A space Y is defined in [3, Definition (2.5), p. 327] to be a strong S^*-space if it is Hausdorff and for each pair of disjoint closed subsets H and K of Y there exist distinct points p and q of Y and a continuous selfmap f of Y such that $f(x) = p$ for $x \in K$. Since X is E-separated and contains a copy of E, it follows readily that X is a strong S^*-space. Consequently, by Theorem (2.7) [3, p. 328] the pair $(S(X), K(X))$ is admissible and (1) now applies. We will use (1) in proving that

$$\mathscr{U}(S(X)) \text{ is } E\text{-compact.} \tag{6}$$

By (1), $\mathscr{U}(S(X))$ is a Hausdorff space which is compact in the usual sense. Thus, if an embedding into a cartesian product of copies of E exists, $\mathscr{U}(S(X))$ must necessarily be embedded as a closed subset. Consequently, we need only prove the existence of an embedding. According to Theorem (2.1) of [4, p. 165], it will be sufficient to show that for each closed subset W of $\mathscr{U}(S(X))$ and each $A \in \mathscr{U}(S(X))$ with $A \notin W$, there exists a continuous function f from $\mathscr{U}(S(X))$ into E and a point $q \in E$ such that $f(B) = q$ for $B \in W$ and $f(A) \neq q$.

Since $\{H(g, \langle y \rangle) : g \in S(X), y \in X\}$ is a subbasis for the closed subsets of $\mathscr{U}(S(X))$, there exists a finite subfamily $\{(g_i, \langle y_i \rangle)\}_{i=1}^N$ of $S(X) \times K(X)$ such that

$$A \notin W^*, \qquad W \subseteq W^* \tag{7}$$

where $W^* = H(g_1, \langle y_1 \rangle) \cup \cdots \cup H(g_N, \langle y_N \rangle)$. Since $A \notin W^*$, there exist by (5), finite subfamilies $\{(h_{i_j}, \langle v_{i_j} \rangle)\}_{j=1}^{N_i}$ of A with the property that $g_i^{-1}(y_i) \cap V_i = \varnothing$ where

$$V_i = h_{i_1}^{-1}(v_{i_1}) \cap \cdots \cap h_{i_{N_i}}^{-1}(v_{i_{N_i}})$$

for $1 \leqq i \leqq N$. Now let $V^* = \bigcap \{V_i\}_{i=1}^N$ and let $H = \bigcup \{g_i^{-1}(y_i)\}_{i=1}^N$. Then $H \cap V^* = \varnothing$ and since X is E-separated and contains a copy E^* of E, there exists a continuous function f mapping X into E^* and two distinct points p and q of E^* such that

$$f(x) = p \quad \text{for } x \in V^* \qquad \text{and} \qquad f(x) = q \quad \text{for } x \in H. \tag{8}$$

As we observed in the discussion preceding (1), the mapping $\mathfrak{f}_f$ defined by $\mathfrak{f}_f(A_{\langle x \rangle}) = A_{f \circ \langle x \rangle}$ is a continuous selfmap of $\mathscr{R}(S(X), K(X))$ which, by (1) has a unique extension to a continuous selfmap $\mathfrak{f}_f^E$ of $\mathscr{U}(S(X))$. Hereafter, we will denote the space $\mathscr{R}(S(X), K(X))$ more simply by $\mathscr{R}(S(X))$. Since X is an S^*-space (in fact, a strong S^*-space), the canonical map $\mathfrak{e}$ which takes $x \in X$ into $A_{\langle x \rangle}$ in $\mathscr{R}(S(X))$

is a homeomorphism from X onto $\mathscr{R}(S(X))$ [3, Theorem (2.3), p. 325]. We assert that

$$\text{(9)} \qquad \mathfrak{f}_f^E \text{ maps } \mathscr{U}(S(X)) \text{ into } \mathfrak{e}[E^*],$$

and

$$\text{(10)} \qquad \mathfrak{f}_f^E(A) = \mathfrak{e}(p) \quad \text{and} \quad \mathfrak{f}_f^E(B) = \mathfrak{e}(q) \quad \text{for } B \in W^*.$$

We recall first of all that f maps all of X into E^*. Hence, for any $A_{\langle x\rangle} \in \mathscr{R}(S(X))$,

$$\mathfrak{f}_f^E(A_{\langle x\rangle}) = A_{\langle f(x)\rangle} = \mathfrak{e}(f(x)) \in \mathfrak{e}[E^*].$$

By (1), $\mathscr{R}(S(X))$ is dense in $\mathscr{U}(S(X))$ and since E^* is compact, we get

$$\mathfrak{f}_f^E[\mathscr{U}(S(X))] = \mathfrak{f}_f^E[\text{cl } \mathscr{R}(S(X))] \subseteq \text{cl } \mathfrak{f}_f^E[\mathscr{R}(S(X))] \subseteq \text{cl } \mathfrak{e}[E^*] = \mathfrak{e}[E^*]$$

where cl denotes closure. This verifies (9). Now we want to show that

$$\text{(11)} \qquad A \in \text{cl}\,\{A_{\langle x\rangle} : x \in V^*\}.$$

Let $\mathscr{C}[H(k_1, \langle r_1\rangle) \cup \cdots \cup H(k_m, \langle r_m\rangle)]$ be any basic open subset of $\mathscr{U}(S(X))$ which contains A. Then by (5), there exist finite subfamilies $\{(t_{i_j}, \langle a_{i_j}\rangle)\}_{j=1}^{M_i}$ of A such that $k_i^{-1}(r_i) \cap U_i = \varnothing$ where

$$U_i = t_{i_1}^{-1}(a_{i_1}) \cap \cdots \cap t_{i_{M_i}}^{-1}(a_{k_{M_i}}).$$

By (4), there exists a point x in $V^* \cap U_1 \cap \cdots \cap U_M$. Thus, $x \notin k_i^{-1}(r_i)$, $i=1, 2, \ldots, M$ from which it follows that $(k_i, \langle r_i\rangle) \notin A_{\langle x\rangle}$, $i=1, 2, \ldots, M$. Therefore,

$$A_{\langle x\rangle} \in \mathscr{C}[H(k_1, \langle r_1\rangle) \cup \cdots \cup H(k_M, \langle r_M\rangle)]$$

and this proves (11). Now for any $x \in V^*$, $f(x)=p$ and we have

$$\mathfrak{f}_f^E(A_{\langle x\rangle}) = A_{\langle f(x)\rangle} = \mathfrak{e}(f(x)) = \mathfrak{e}(p).$$

This fact, together with (11) implies that $\mathfrak{f}_f^E(A)=\mathfrak{e}(p)$ which is the first half of (10). In much the same way that we verified (11), one can show that if $B \in W^*$, then $B \in \text{cl}\,\{A_{\langle x\rangle} : x \in H\}$ and since $f(x)=q$ for $x \in H$, it follows that $\mathfrak{f}_f^E(A_{\langle x\rangle})=\mathfrak{e}(q)$ for each x in H. Therefore, $\mathfrak{f}_f^E(B)=e(q)$ for each $B \in W^*$ and this completes the proof of statement (10). In view of the discussion immediately following statement (6), it is a consequence of (9) and (10) that (6) is valid, that is, $\mathscr{U}(S(X))$ is E-compact.

Now we are in a position to show that $\mathscr{U}(S(X))$ is $\beta_E X$. Actually, we show that $\mathscr{U}(S(X))$ is $\beta_E\mathscr{R}(S(X))$ but since the canonical map $\mathfrak{e}$ maps X homeomorphically onto $\mathscr{R}(S(X))$ we identify the two spaces. In order to conclude that $\mathscr{U}(S(X))$ is $\beta_E\mathscr{R}(S(X))$ it is sufficient, according to Theorem 4.14 of [4, p. 177] to show that $\mathscr{U}(S(X))$ is E-compact and also that every continuous function from $\mathscr{R}(S(X))$ into E can be continuously extended to a function which maps $\mathscr{U}(S(X))$ into E. We have yet to verify the latter and for this, it will be sufficient to show that any continuous function f from $\mathscr{R}(S(X))$ into $\mathrm{e}[E^*]$ has a continuous extension to a function which maps $\mathscr{U}(S(X))$ into $\mathrm{e}[E^*]$. To get this extension we note that $g=\mathrm{e}^{-1}\circ f\circ \mathrm{e}$ belongs to $S(X)$ and hence $\mathfrak{f}_g^E$ is a continuous selfmap of $\mathscr{U}(S(X))$ by (1). For any $A_{\langle x\rangle}\in\mathscr{R}(S(X))$, we have

$$\mathfrak{f}_g^E(A_{\langle x\rangle}) = A_{g\circ\langle x\rangle} = A_{\langle g(x)\rangle} = \mathrm{e}(g(x)) = f(\mathrm{e}(x)) = f(A_{\langle x\rangle}).$$

Thus $\mathfrak{f}_g^E$ is indeed an extension of f which (since $\mathrm{e}[E^*]$ is compact and $\mathscr{R}(S(X))$ is dense in $\mathscr{U}(S(X))$) maps $\mathscr{U}(S(X))$ into $\mathrm{e}[E^*]$. This completes the proof of the Main Theorem.

If, in the Main Theorem, we take E to be the closed unit interval, we immediately get the following result which first appeared in [3, p. 329] as Corollary (2.8).

COROLLARY 1. *Suppose X is normal, Hausdorff and contains an arc. Then the $\mathscr{K}$-structure space of $S(X)$ is the Stone-Čech compactification of X.*

A partition of a space X is any finite collection of mutually disjoint subsets of X which are both closed and open and whose union is all of X. A 0-dimensional space here will mean a space whose Lebesgue dimension is zero, that is, one with the property that every open cover has a refinement by a partition of the space.

COROLLARY 2. *Let X be a normal 0-dimensional Hausdorff space. Then the $\mathscr{K}$-structure space of $S(X)$ is the Stone-Čech compactification of X.*

Proof. Here again we apply the Main Theorem and in this case we take E to be the two-point discrete space $\mathscr{D}$. The conclusion is immediate if X has only one point so we assume that X has more than one point and, consequently contains a copy of $\mathscr{D}$. To show that X is $\mathscr{D}$-separated, let H and K be two disjoint closed subsets of X. Then $\{\mathscr{C}H, \mathscr{C}K\}$ is a cover of X and hence has a refinement by a partition $\{V_i\}_{i=1}^N$ of X.

Let $W=\bigcup\{V_i : V_i\subset\mathscr{C}H\}$. Then W is a subset of X which is both closed and open. Furthermore, $H\subset\mathscr{C}W$ and since $K\subset\mathscr{C}H$ and $\{V_i\}_{i=1}^N$ is a refinement of $\{\mathscr{C}H, \mathscr{C}K\}$ it readily follows that $K\subset W$. Therefore, if p and q denote the two points of $\mathscr{D}$, the function which maps all of W into p and $\mathscr{C}W$ into q is continuous and we conclude that X is $\mathscr{D}$-separated. Then by the Main Theorem, the $\mathscr{D}$-structure space of $S(X)$ is $\beta_{\mathscr{D}}X$, the largest $\mathscr{D}$-compactification of X. Now it is well known that the Stone-Čech compactification βX of X is the largest among all the compactifications of X. So, in order to conclude that $\beta_{\mathscr{D}}X=\beta X$, it is sufficient to observe that βX is a $\mathscr{D}$-compactification of X. In [2, p. 243], a modified definition of Lebesgue dimension is used. However, the definition there agrees with the usual one for normal spaces. Consequently, it follows from Theorem 16.11 of [2, p. 245] that βX is 0-dimensional. Then βX is also 0-dimensional in the sense of [4], that is, it has a basis of sets which are both open and closed. But this implies that βX is $\mathscr{D}$-compact [4, p. 176] and hence that βX is a $\mathscr{D}$-compactification of X.

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